STATIONARY ISOTHERMIC FLOW OF A NON-NEWTONIAN LIQUID IN A PARABOLIC CONVERGENT CHANNEL

B. I. Lapushina and A. Kh. Kim

UDC 532.135

An approximate solution of the problem of stationary isothermic flow of a non-Newtonian liquid in a parabolic convergent channel is given for the condition $r \ll z$. Expressions for the velocity distribution are derived.

The motion of a non-Newtonian liquid in a circular cone whose side surface is formed by rotation of the parabola $r = az^2$ about the z axis is considered. The rheological behavior of the moving system is described by the equation

$$\Pi_0 = 2kh^{n-1}\dot{\Phi}_0. \tag{1}$$

It is assumed that the motion is stationary and isothermic. The problem is solved for a convergent channel with infinite length which satisfies the condition

$$r \ll z \ (a \ll 1). \tag{2}$$

This condition is encountered in designing nozzles for atomization, mechanical deposition of coatings, etc.

As it moves in a parabolic convergent channel, the system experiences both linear and angular deformations. In this case [1], the shear extends to the whole region. The flow is convergent. The differential equation of motion is obtained by solving simultaneously Eq. (1) and the Cauchy equilibrium equation

$$\operatorname{div} \Pi = \rho_0 \bar{a}. \tag{3}$$

Since the flow is axisymmetric,

$$v_r = v_r(r,\ z);\ v_z = v_z(r,\ z);\ v_\varphi = 0. \tag{4}$$

A series of isotropic parabolas $r = \nu az^2$, where $0 \le \nu \le 1$, can be drawn inside the convergent channel (see Fig. 1). In solving the problem, we substitute the new coordinates ν and z for r and z and write the differential equations in terms of the new variables, allowing for the relationships between the new and the old coordinates.

We introduce the auxiliary functions

$$\begin{aligned} \varphi_1 &= 2kh^{n-1}\dot{e}_{rr}, \\ \varphi_2 &= 2kh^{n-1}\dot{e}_{zz}, \\ \tau &= 2kh^{n-1}\dot{e}_{rz}. \end{aligned} \tag{5}$$

Then, the differential equations of motion are written in the following form:

$$\begin{aligned} &-\frac{\partial p}{\partial \nu} + \frac{\partial \varphi_1}{\partial \nu} + az^2 \frac{\partial \tau}{\partial z} - 2\nu az \frac{\partial \tau}{\partial \nu} \\ &+ \frac{2\varphi_1 + \varphi_2}{\nu} = \rho_0 az^2 \frac{dv_r}{dt}, \end{aligned} \tag{6}$$

Belorussian Polytechnic Institute, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 20, No. 2, pp. 205-209, February, 1971.

$$-\frac{\partial p}{\partial z}+\frac{\partial \varphi_2}{\partial z}+\frac{2\nu}{z}\cdot\frac{\partial p}{\partial \nu}-\frac{2\nu}{z}\cdot\frac{\partial \varphi_2}{\partial \nu}$$

$$+\frac{1}{az^2}\cdot\frac{\partial \tau}{\partial \nu}+\frac{\tau}{\nu a z^2}=\rho_0\frac{dv_z}{dt}. \tag{7}$$

After eliminating p from these equations, we obtain

$$\frac{\partial^2\varphi_1}{\partial\nu\partial z}-\frac{2}{z}\cdot\frac{\partial^2\varphi_1}{\partial\nu^2}+\frac{2}{\nu}\cdot\frac{\partial\varphi_1}{\partial z}-\frac{6}{z}\cdot\frac{\partial\varphi_1}{\partial\nu}+\frac{1}{\nu}\cdot\frac{\partial\varphi_2}{\partial z}$$

$$-\frac{\partial^2\varphi_2}{\partial\nu\partial z}+\frac{2\nu}{z}\cdot\frac{\partial^2\varphi_2}{\partial\nu^2}+az^2\frac{\partial^2\tau}{\partial z^2}-4\nu az\frac{\partial^2\tau}{\partial\nu\partial z}$$

$$+\left(4\nu^2 a-\frac{1}{az^2}\right)\frac{\partial^2\tau}{\partial\nu^2}+\left(6\nu a-\frac{1}{\nu az^2}\right)\frac{\partial\tau}{\partial\nu}$$

$$+\frac{\tau}{\nu^2az^2}+\rho_0\left[\frac{\partial}{\partial\nu}\left(\frac{dv_z}{dt}\right)+2\nu az\frac{\partial}{\partial\nu}\left(\frac{dv_r}{dt}\right)\right.$$

$$\left.-az^2\frac{\partial}{\partial z}\left(\frac{dv_r}{dt}\right)\right]=0. \tag{8}$$

We introduce the stream function which satisfies the continuity equation,

$$\psi(\nu, z)=\sum_{i=0}^{\infty}\frac{\omega_i}{z^i}, \tag{9}$$

where $\omega_i = \omega_i(\nu)$. Such a stream function was used in considering the motion of a viscous liquid in a cone [2] and a viscoplastic medium in a flat parabolic divergent channel [1].

The velocity distribution was determined mainly by the first term of the expansion – the basic function [1]. The correction introduced by the connecting functions is negligible. In the first approximation, the streamlines follow the isotropic curves.

Retaining only the first term of the expansion in determining the basic function, we obtain

$$v_r=\frac{2\omega_0'}{az^3};\quad v_z=\frac{\omega_0'}{\nu a^2z^4}. \tag{10}$$

In order to satisfy the condition at the axis ($v_r|_{\nu=0}=0$; $v_z|_{\nu=0}\neq 0$), we seek ω_0' in the form $\omega_0'=\nu f(\nu)$. Then,

$$v_r=\frac{2\nu f}{az^3};\quad v_z=\frac{f}{a^2z^4}. \tag{11}$$

After calculating the derivatives of the functions figuring in the differential equation (8), we reach the conclusion that, with the assumption (2) for a mildly sloping convergent channel, the terms containing φ_1, φ_2, and their derivatives in the differential equation can be neglected, and the intensity of the deformation rates can be determined by means of

$$h=\frac{f'}{a^3z^6}. \tag{12}$$

If $r \ll z$, we obtain the following equation for determining the basic function:

$$\nu^2\frac{\partial^2\tau}{\partial\nu^2}+\nu\frac{\partial\tau}{\partial\nu}-\tau=0, \tag{13}$$

where $\tau = kh^n$, since $\tau > 0$ for a convergent flow. By solving this equation, we determine the value of the function f,

$$f=f_0\left(1-\nu^{\frac{1}{n}+1}\right). \tag{14}$$

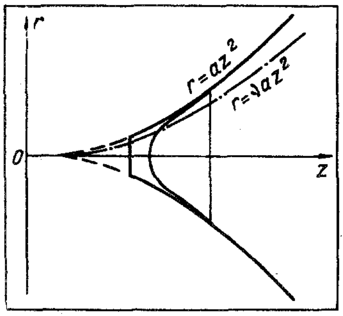

Fig. 1. Motion in a parabolic convergent channel.

The integration constants are determined from the following equations:

1) the adhesion condition,

$$v_r|_{\nu=1} = 0,\ v_z|_{\nu=1} = 0,$$

whence $f|_{\nu=1} = 0$;

2) the condition of discharge constancy,

$$2\pi \int_0^1 \omega(\nu)\, d\nu = -Q,$$

$$3)\ v_z|_{\nu=0} = \frac{f_0}{a^2 z^4},\ \text{ i.e. }\ f|_{\nu=0} = f_0. \tag{15}$$

By substituting the value of f in Eqs. (11), we obtain the expressions describing the velocity distribution in the first approximation:

$$v_r = \frac{2\nu f_0}{az^3}\left(1 - \nu^{\frac{1}{n}+1}\right),$$
$$v_z = \frac{f_0}{a^2 z^4}\left(1 - \nu^{\frac{1}{n}+1}\right). \tag{16}$$

In the flow of pseudo plastics (n < 1) and, in the limiting case, Newtonian liquids (n = 1), inertial forces do not affect the value of the basic function. The flow of a Newtonian liquid in a parabolic convergent channel constitutes a particular case of the problem considered above. We have analyzed this problem for the condition $r \ll z$. The following equation was obtained for determining the basic function in this case:

$$\omega_0^{IV} \nu^3 - 2\omega_0''' \nu^2 + 3\omega_0'' \nu - 3\omega_0' = 0. \tag{17}$$

By solving (17) for the boundary conditions, we obtain the expressions

$$v_r = \frac{2\nu f_0}{az^3}(1 - \nu^2),$$
$$v_z = \frac{f_0}{a^2 z^4}(1 - \nu^2), \tag{18}$$

which in fact can be obtained from (16) for n = 1. As was mentioned earlier, expressions (16) and (18) provide an idea of the velocity profile in the first approximation. By determining the correcting functions, we can obtain a solution with any degree of accuracy.

In determining the first correcting function, we use the first two terms of the expansion defined by (9):

$$\psi(\nu, z) = \omega_0 + \frac{\omega_1}{z}. \tag{19}$$

Then,

$$v_r = \frac{2\omega_0'}{az^3} + \frac{\omega_1 + 2\nu\omega_1'}{\nu a z^4},$$
$$v_z = \frac{\omega_0'}{\nu a^2 z^4} + \frac{\omega_1'}{\nu a^2 z^5}, \tag{20}$$

the first terms of which have already been found in determining the basic function.

In order to satisfy the condition at the axis ($v_r|_{\nu=0} = 0$, $v_z|_{\nu=0} \neq 0$), we seek ω_1 in the following form:

$$\omega_1 = \nu^2 \varphi(\nu). \tag{21}$$

Then,

$$v_r = \frac{2\nu f_0}{az^3}(1-\nu^{\frac{1}{n}+1}) + \frac{1}{az^4}(5\nu\varphi + 2\nu^2\varphi'),$$
$$v_z = \frac{f_0}{a^2z^4}(1-\nu^{\frac{1}{n}+1}) + \frac{1}{a^2z^5}(2\varphi + \nu\varphi'). \qquad (22)$$

In determining the correcting function, we must decide which one will bring inertial forces into the equation for each specific case in dependence on the deviation of n from unity.

Neglecting small quantities and assuming that the deviation from Newtonian behavior is such that inertial forces do not affect the first correcting function, we obtain a differential equation, the solution of which provides the value of φ in the following form:

$$\varphi = \frac{C_1 n^2 \nu^{\frac{1}{n}+1}}{(3n+1)(n+1)} + C_2. \qquad (23)$$

The function φ must satisfy the following conditions:

$$\varphi|_{\nu=1}=0; \ \varphi|_{\nu=0}=\varphi_0,$$

whence

$$C_1 = 0; \ C_2 = 0.$$

Thus, the first correcting function vanishes. The correcting functions are equal to zero whenever the differential equations used for their determination are homogeneous. The function which is determined by taking into account inertial forces is the first one that is different from zero. The smaller the value of n, the larger the index number of the function whose value is affected by inertial forces. For a Newtonian liquid (n = 1), even the first correcting function must be determined by taking into account inertial forces. For n > 1 (dilatant liquids), inertial forces can affect the basic function, beginning with certain values of the index n.

The pressure necessary for the motion of a non-Newtonian liquid in the convergent channel can be determined by integrating Eqs. (6) and (7).

NOTATION

Π_0 is the stress tensor deviator;
$\dot{\Phi}_0$ is the deviator of the deformation rate tensor;
h is the intensity of deformation rates;
k is the consistency measure;
n is the index of deviation from Newtonian behavior;
ρ_0 is the density;
τ is the shearing stress;
$\dot{e}_{ik}$ are the components of the deformation rate tensor;
Q is the discharge.

LITERATURE CITED

1. A. Kh. Kim, Author's Abstract of Dissertation, Belorussian Polytechnic Institute, Minsk (1966).
2. N. A. Slezkin, Matematicheskii Sbornik, 42 (1), 43 (1935).